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Third-Order Constants of Motion in Quantum Mechanics

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Abstract

We investigate the general form of a third-order linear differential operator that is required to commute with the Schrödinger Hamiltonian in two dimensions, and find that the third-order part must be a polynomial of third degree in the generators of the Euclidean group. Partial differential equations that the potential V must satisfy are derived, and solved for the special cases where the Schrödinger equation separates in polar or Cartesian coordinates. The functions V thus obtained are nonsingular, but are periodic through elliptic functions. After separation of variables, the Schrödinger equation gives Lame's equation.

1. *Introduction*

Following the work of Stackel (1891) and Robertson (1928), the conditions for the separability of Schrödinger's equation were investigated (Eisenhart, 1934). Using the powerful methods of Riemannian geometry, Eisenhart found that, if the potential energy has appropriate forms, the Schrödinger equation separates in the same coordinate systems as the Helmholtz equation: four systems in the two-dimensional case and eleven in the three-dimensional case. The forms of the potential energy (for one particle) were also obtained (Eisenhart, 1948) both in curvilinear and Cartesian coordinates. It is interesting to note that, many years later, a search for constants of the motion in nonrelativistic quantum mechanics led to the same separation problem and the same forms of the potential energy. Considering only operators of second degree in the momenta, it was found (Fris et al., 1965; Winternitz et al., 1966) that such quadratic constants of the motion can exist in the two-dimensional case if and only if there is an orthogonal coordinate system in which the variables separate. In the three-dimensional case, the same condition of separability is equivalent to the existence of two quadratic constants of the motion (Makarov et al., 1967). In the separated Schrödinger equation the

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potential energy part appears as a sum of arbitrary functions of individual coordinates. The arbitrariness is partly or wholly removed if the equation separates in two or more coordinate systems, either of different types, or of the same type but translated or rotated with respect to one another. Some of the terms in the potential energy so determined become singular on entire lines and surfaces and are therefore not of much physical interest. But the main result, namely, the connection of separability with the existence of quadratic constants of the motion, is certainly of interest, and has prompted the present authors to undertake an investigation on higher-order constants of the motion.

To understand the nature of the problem and to have an idea of the difficulties that are likely to arise in such an investigation we consider, in the two-dimensional case, a third-order operator of the type

$$\begin{aligned} L = {} & \phi_1 p_1^3 + \phi_2 p_1^2 p_2 + \phi_3 p_1 p_2^2 + \phi_4 p_2^3 \\ & + \phi_5 p_1^2 + \phi_6 p_1 p_2 + \phi_7 p_2^2 + \phi_8 p_1 + \phi_9 p_2 + \phi \end{aligned} \tag{1.1}$$

where $p_1 = \partial/\partial x, p_2 = \partial/\partial y$, and the ϕ's are functions of x and y. Following the procedure adopted previously (Winternitz et al., 1966) we evaluate the commutator $[L, H]$ [with $H = -\frac{1}{2}(p_1^2 + p_2^2) + V = -\frac{1}{2}\Delta + V$] and obtain a system of partial differential equations for the determination of the unknown ϕ's and V. As a first step towards a solution we eliminate the ϕ's and obtain two equations, of the second and the third order, in V alone. These equations possess high symmetry and, properly handled, are likely to yield interesting results. Postponing a more detailed study of the equations to a future date we consider here two simple special cases in which most of the constants appearing in the equations are set equal to zero. The variables separate in polar coordinates in the first case and in Cartesian coordinates in the second case. The separated equation for the potential, in both the cases, has a solution of the form $V = A + B\,\mathrm{sn}^2(u, k)$ $(u = \alpha x, \beta y, \gamma\theta)$, and the lowest eigenfunctions are $\mathrm{sn}(u, k)$, $\mathrm{cn}(u, k)$, $\mathrm{dn}(u, k)$ with periods $4K$ or $2K$, where K is the complete elliptic integral of the first kind. We do not consider here other periodic solutions and quasiperiodic solutions of Bloch's type. By making k approach unity one can also obtain a Hulthen-like potential, and the corresponding eigenfunctions. When the expressions for the ϕ's are substituted in (1.1), L is found to be irreducible and to have the eigenvalue zero for nondegenerate periodic functions.

2. *The Differential Equations*

Evaluating the commutator $[L, H]$ and separately equating to zero the coefficients of the various powers of p_1 and p_2 we have the following system of equations for the determination of $\phi_1, \ldots, \phi_7$:

$$\phi_{1x} = 0, \phi_{1y} = -\phi_{2x}, \phi_{2y} = -\phi_{3x}, \phi_{3y} = -\phi_{4x}, \phi_{4y} = 0 \tag{2.1}$$

$$\begin{aligned} \Delta\phi_1 &= -2\phi_{5x}, \Delta\phi_2 = -2\phi_{5y} - 2\phi_{6x} \\ \Delta\phi_3 &= -2\phi_{6y} - 2\phi_{7x}, \Delta\phi_4 = -2\phi_{7y} \end{aligned} \tag{2.2}$$

where $\phi_{1x} = \partial\phi_1/\partial x$, etc. From the structure of the equations (2.1) it is evident that ϕ_1 is a cubic in y only and ϕ_4 is a cubic in x only. Once the forms of ϕ_1 and ϕ_4 are known the expressions for the other ϕ's can be written down by inspection. Thus

$$\begin{aligned} \phi_1 &= ay^3 + by^2 + cy + d \\ \phi_2 &= -3axy^2 - 2bxy - cx - fy^2 - hy - k \\ \phi_3 &= 3ax^2y + bx^2 + 2fxy + hx + gy + l \\ \phi_4 &= -ax^3 - fx^2 - gx - e \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \phi_5 &= -3axy - bx + my^2 + (q + f)y + r \\ \phi_6 &= 3ax^2 - 2mxy - 3ay^2 + (n - b)y - qx + s \\ \phi_7 &= 3axy + mx^2 - nx + fy - r \end{aligned} \tag{2.4}$$

The constants appearing in these expressions are all arbitrary.

Previously (Winternitz et al., 1966) it was shown that the most general differential operator that may appear in a second-order constant of the motion consists of an arbitrary symmetric polynomial of degree 2 in the generators of the Euclidean group, i.e., in p_1, p_2 and $M = \partial/\partial\theta$ where θ is the polar angle. Substituting (2.3) and (2.4) into (1.1) gives the extension of this result to the third-order case:

$$\begin{aligned} (L - \phi_8 p_1 - \phi_9 p_2 - \phi) = {}& -aM^3 + dp_1{}^3 - ep_2{}^3 - kp_1{}^2p_2 + lp_1p_2{}^2 \\ & + \tfrac{1}{2}h(p_2Mp_1 + p_1Mp_2) - cp_1Mp_1 - gp_2Mp_2 \\ & + bMp_1M - fMp_2M + (\text{polynomial of degree 2} \\ & \text{in } p_1, p_2 \text{ and } M) \end{aligned} \tag{2.5}$$

(Note: first-order terms are absorbed by redefining ϕ_8 and ϕ_9.)

We are now faced with the much harder task of determining the remaining functions ϕ_8, ϕ_9, ϕ, and V which satisfy the system of linked equations

$$\begin{aligned} -3\phi_1 V_x - \phi_2 V_y &= \phi_{8x} + m \\ -2\phi_2 V_x - 2\phi_3 V_y &= \phi_{8y} + \phi_{9x} \\ -\phi_3 V_x - 3\phi_4 V_y &= \phi_{9y} + m \end{aligned} \tag{2.6}$$

$$\begin{aligned} 3\phi_1 V_{xx} + 2\phi_2 V_{xy} + \phi_3 V_{yy} + 2\phi_5 V_x + \phi_6 V_y &= -\phi_x - \tfrac{1}{2}\Delta\phi_8 \\ \phi_2 V_{xx} + 2\phi_3 V_{xy} + 3\phi_4 V_{yy} + \phi_6 V_x + 2\phi_7 V_y &= -\phi_y - \tfrac{1}{2}\Delta\phi_9 \end{aligned} \tag{2.7}$$

$$LV = H\phi \tag{2.8}$$

Elimination of ϕ_8, ϕ_9, and ϕ from (2.6) and (2.7) gives

$$(\phi_{3y} + \phi_{2x} - \phi_6)V_{xx} + (\phi_{3x} - \phi_{2y} + 2\phi_5 - 2\phi_7)V_{xy} + (\phi_6 - \phi_{2x} - \phi_{3y})V_{yy} + (2\phi_{5y} - \phi_{6x} - \phi_{2yy} + \phi_{3xy})V_x + (\phi_{6y} - 2\phi_{7x} - \phi_{2xy} + \phi_{3xx})V_y = 0 \tag{2.9}$$

and eliminating ϕ_8 and ϕ_9 from (2.6) only gives

$$(2\phi_2 - 3\phi_4)V_{xxy} + (2\phi_3 - 3\phi_1)V_{xyy} - \phi_2 V_{yyy} - \phi_3 V_{xxx} + 2(\phi_{2y} - \phi_{3x})V_{xx} + 2(\phi_{3y} - 3\phi_{4x} - 3\phi_{1y} + \phi_{2x})V_{xy} + 2(\phi_{3x} - \phi_{2y})V_{yy} + (2\phi_{2xy} - 3\phi_{1yy} - \phi_{3xx})V_x + (2\phi_{3xy} - \phi_{2yy} - 3\phi_{4xx})V_y = 0 \tag{2.10}$$

The second-order equation (2.9) has the same form as that previously obtained by eliminating ϕ (Winternitz et al., 1966), but the third-order equation (2.10) is new. To determine ϕ_8, ϕ_9, ϕ, and V it is best to start from (2.9) and (2.10). One may either seek a common solution of the two equations or choose the constants in such a manner that one of the equations is trivially satisfied (i.e., reduces to $0 = 0$) leaving the other to give conditions on V. The function V obtained in this way may be substituted into (2.6) and (2.7) and ϕ_8, ϕ_9, ϕ are determined. Finally the remaining constraint (2.8) on V must be investigated.

3. *A Solution in Polar Coordinates*

To obtain a particular solution with the Schrödinger equation separable in polar coordinates we set equal to zero all the constants in equations (2.3) and (2.4) except m and a. Then equations (2.9) and (2.10) have a common solution of the form

$$V(r, \theta) = F(r) + r^{-2}G(\theta) \tag{3.1}$$

where F and G are arbitrary functions. With these specializations, and also setting $a = \frac{1}{3}$, equations (2.6) become

$$\phi_{8x} = -m + \sin^2\theta\, G'(\theta), \qquad \phi_{9y} = -m + \cos^2\theta\, G'(\theta)$$

$$\phi_{8y} + \phi_{9x} = -2\sin\theta\cos\theta\, G'(\theta) \tag{3.2}$$

(2.8) becomes

$$-\tfrac{1}{3}(V_{\theta\theta\theta} + V_\theta) + mV_{\theta\theta} + mrV_r = -\tfrac{1}{2}\Delta\phi - \phi_8 V_x - \phi_9 V_y \tag{3.3}$$

while (2.7) are equivalent to

$$G'(\theta) = -r\phi_r - \tfrac{1}{2}x\Delta\phi_8 - \tfrac{1}{2}y\Delta\phi_9$$

$$G''(\theta) - 2mG'(\theta) = \phi_\theta - \tfrac{1}{2}y\Delta\phi_8 + \tfrac{1}{2}x\Delta\phi_9 \tag{3.4}$$

Equations (3.2) are satisfied by

$$\phi_8 = -mr\cos\theta - r\sin\theta\, G(\theta)$$

$$\phi_9 = -mr\sin\theta + r\cos\theta\, G(\theta) \qquad (3.5)$$

A more general solution of (3.2) contains three arbitrary constants, but including these turns out to give the same potentials, either relative to rotated axes, or with an additive constant, or with G differing by a constant [this difference can be absorbed in F in (3.1)].

Finally, substituting (3.5) into (3.4) gives

$$\phi_r = 0, \qquad \phi_\theta = \tfrac{1}{2}G'' - 2mG' \qquad (3.6)$$

while (3.3) becomes

$$\tfrac{1}{4}G'^2 = G^3 - G^2 + \beta G + \gamma \qquad (3.7)$$

with arbitrary β and γ.

Equation (3.7) leads to an elliptic integral and has nonsingular, periodic solutions of the form

$$G = -a + (a - b)\,\mathrm{sn}^2(u, k) \qquad (3.8)$$

with $k^2 = (a - b)/(a - c)$ and $u = \theta(a - c)^{1/2}$. The three constants a, b, and c must satisfy $a + b + c = -1$, $a > b > c$, and one further condition to make G have period 2π: $\pi(a - c)^{1/2}$ must be an integer multiple of

$$K = \int_0^{\pi/2} dt(1 - k^2\sin^2 t)^{-1/2} \qquad (3.9)$$

Note that the radial part F of V remains completely arbitrary. The constant of the motion is given by (2.5), (3.5), and (3.6):

$$L = M^3 + (1 - 3G)M - \tfrac{3}{2}G' \qquad (3.10)$$

When the Schrödinger equation is separated in polar coordinates, then (3.1) and (3.8) give Lamé's equation for the angular part. There are three non-degenerate angular functions, namely, $\mathrm{dn}(u, k)$, $\mathrm{cn}(u, k)$, and $\mathrm{sn}(u, k)$. Since L converts odd functions into even functions, and vice versa, L must give zero when operating on these three functions. The angular equation will also have an infinite, discrete set of eigenvalues which are twofold degenerate (Ince, 1940).

4. *A Solution in Cartesian Coordinates*

The Cartesian case is very similar to the polar case. To get a separable solution for V we now put all the constants except for d, e, and r equal to zero in equations (2.6)-(2.10). Then (2.9) and (2.10) are satisfied by

$$V(x, y) = f(x) + g(y) \qquad (4.1)$$

for arbitrary f and g, and (2.6) and (2.7) give

$$\phi_8 = -3df(x) - \alpha y + \mu, \qquad \phi_9 = 3eg(y) + \alpha x + \nu$$

$$\phi = 2r\{g(y) - f(x)\} + \tfrac{3}{2}\{eg'(y) - df'(x)\} \tag{4.2}$$

Finally, (2.8) is variable separable only if $\alpha = 0$, in which case f and g satisfy equations like (3.7), and the new results that may be obtained are essentially one-dimensional. Corresponding to (3.8) and (3.10), we have that the potential

$$f(x) = -a + (a - b)\mathrm{sn}^2(u, k) \tag{4.3}$$

with $k^2 = (a - b)/(a - c)$ and $u = x(a - c)^{1/2}$, where $a > b > c$, allows

$$L = p_1^3 - 3f(x)p_1 - \tfrac{3}{2}f'(x) - (a + b + c)p_1 \tag{4.4}$$

as a constant of the motion. As in the angular case, the one-dimensional Schrödinger equation is Lamé's equation, and L is zero in the lowest three energy eigenfunctions $\mathrm{dn}(u, k)$, $\mathrm{cn}(u, k)$ and $\mathrm{sn}(u, k)$, which have the energies $-\frac{1}{2}a - \frac{1}{2}b$, $-\frac{1}{2}a - \frac{1}{2}c$, and $-\frac{1}{2}b - \frac{1}{2}c$, respectively. The one-dimensional problem will have an infinite, discrete set of higher-energy eigenvalues, which are two-fold degenerate (Ince, 1940).

In contrast to the angular potential, where (3.9) must be satisfied to make the period 2π, equation (4.3) allows any period. By letting $k^2 \to 1$, the non-periodic potential $V(x) = -\gamma^2 \mathrm{sech}^2(\gamma x)$ is obtained, and the constant of the motion becomes $L = p_1^3 - \gamma^2 p_1 - 3Vp_1 - \frac{3}{2}V'$. This gives zero when acting on the even bound state wave function $\mathrm{sech}(\gamma x)$, or the odd zero-energy wave function $\tanh(\gamma x)$. However, at the positive energy $E = \frac{1}{2}p^2$, the even and odd parity wave functions are not the eigenfunctions of L, which are $e^{\pm ipx}(p \pm i\gamma \tanh \gamma x)$ belonging to the eigenvalues $\mp ip(\gamma^2 + p^2)$.

Two other special cases of equations (2.6)-(2.10) have been studied with the assumption (4.1). Taking k and l nonzero in addition to d, e, and r leads to either $f(x) = 0$ or $g(y) = 0$ and the trivial constants p_2^3 or p_1^3. Taking c, g, and h to be nonzero as well, and excluding potentials singular along a line, only gives $V = \alpha(x^2 + y^2) + \beta x + \gamma y$, which is not expected to yield any new results of interest.

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